6-1-2010

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Recommended Citation

Ding, J., Tang, Y. (2010). Non-Convexity of the Dimension Function for Sierpinski Pedal Triangles. *Fractals-Complex Geometry Patterns and Scaling in Nature and Society, 18*(2), 191-195.
Available at: https://aquila.usm.edu/fac_pubs/758

NON-CONVEXITY OF THE DIMENSION FUNCTION FOR SIERPIŃSKI PEDAL TRIANGLES

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Abstract

We disprove the conjecture of the paper [5] on the Schur-convexity of the dimension function for the family of Sierpiński pedal triangles. We also show that this function is not convex and the related area-ratio function is not concave in their respective domain.

Keywords: Sierpiński Pedal Triangles; Sierpiński Triangle; Moran Equation; Fractal Dimension; Schur-convex Functions; Convex Functions.

1 INTRODUCTION

In the paper [5], a family of self-similar fractals called the *Sierpiński pedal triangles* were constructed, and some of their basic properties have been obtained. Given any acute triangle $\triangle A_0B_0C_0$, the corresponding Sierpiński pedal triangle is formed by deleting the pedal triangle at each step of the construction, analogous to the definition of the classic Sierpiński triangle. More specifically, we draw the *pedal triangle* $\triangle A_1B_1C_1$ of $\triangle A_0B_0C_0$ by connecting the three feet of altitudes of $\triangle A_0B_0C_0$, and then delete the interior of $\triangle A_1B_1C_1$ from $\triangle A_0B_0C_0$. The remaining part is the union of three triangles $\triangle A_0B_1C_1, \triangle A_1B_0C_1$, and $\triangle A_1B_1C_0$, which are all similar to $\triangle A_0B_0C_0$. For each of them we repeat the same procedure, and the remaining set is the union of 3^2 similar triangles. This procedure is continued to infinity and the Sierpiński pedal triangle is just the limiting set of such nested subsets of $\triangle A_0B_0C_0$. If the initial triangle $\triangle A_0B_0C_0$ is a right one, then its pedal triangle $\triangle A_1B_1C_1$ is degenerated into a line segment, and the resulting Sierpiński pedal triangle can still be constructed in the same way.

Let x and y denote two angles of the initial triangle $\triangle A_0B_0C_0$ of the Sierpiński pedal triangle that will be denoted as $\mathrm{SPT}(x, y)$. Because of the self-similarity property of this fractal associated with the three affine constraction constants $\cos x, \cos y$, and $\cos z$ with $z = \pi - x - y$, the fractal dimension $d = d(x, y)$ of $\mathrm{SPT}(x, y)$ is the unique solution of the so-called *Moran equation* [1]

$$\cos^d x + \cos^d y + \cos^d z = 1. \tag{1}$$

It is clear that $d(x, y) = d(y, z) = d(z, x)$ whenever $x+y+z = \pi$. The dimension function $d(x, y)$ is also a symmetric function since $d(x, y) = d(y, x)$. Note that $d(\pi/3, \pi/3) = \ln 3/ \ln 2$ and $d(\pi/2, y) = d(x, \pi/2) = d(x, \pi/2 - x) = 2$.

Properties of $d(x,y)$ were first studied in [5]. For example, it was shown there that the dimension of the Sierpiński triangle is a strict local minimum of the dimension function for Sierpiński pedal triangles. It was conjectured in the same paper that $\ln 3/\ln 2$ be the global minimum of the dimension function. This conjecture has been proved recently in [2].

Denote by

$$I = \left\{(x,y) \in \mathbb{R}^2 : 0 < x, y < \frac{\pi}{2}, \ \frac{\pi}{2} < x + y < \pi\right\}$$

the open triangular region, which is called the *index domain* that represents all acute triangles. Note that the boundary of I represents all right triangles. Based on the fact that the *area-ratio function*

$$r(x,y) = -2\cos x \cos y \cos(x+y), \quad \forall\ (x,y) \in I, \tag{2}$$

which gives the ratio of the area of the pedal triangle $\triangle A_1B_1C_1$ and the area of its "mother" triangle $\triangle A_0B_0C_0$ with two angles x and y, is Schur-concave, it was further conjectured in [5] that the dimension function d be a Schur-convex function of (x,y) in its domain. The purpose of this paper is to *disprove* the above conjecture.

As another interesting result, in the next section we first prove that the area-ratio function is not a concave one. Then in Section 3 we provide a negative answer to the last conjecture of [5] on the Schur-convexity of the dimension function for Sierpiński pedal triangles. It will further be shown that this function is not convex, either. We conclude in Section 4.

2 NON-CONCAVITY OF THE AREA-RATIO FUNCTION

A real square matrix S is said to be *doubly stochastic* if it is nonnegative and each of its row sums and column sums is 1. Thus, a 2×2 doubly stochastic matrix can be written as

$$S = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}, \quad p \in [0,1]. \tag{3}$$

A real-valued function f of n variables defined on a region D of $\mathbb{R}^n$ is said to be *Schur-convex* if for all $n \times n$ doubly stochastic matrices S.

$$f(S\mathbf{x}) \leq f(\mathbf{x}), \quad \forall\, \mathbf{x} \in D. \tag{4}$$

When D is a plane region, from the expression (3) of 2×2 doubly stochastic matrices, f is Schur-convex if and only if for all $0 \leq p \leq 1$,

$$f(px + (1-p)y, (1-p)x + py) \leq f(x,y), \quad \forall\, (x,y) \in D.$$

Here, we assume that the domain D is convex and satisfies the property that $S\mathbf{x} \in D$ for all $\mathbf{x} \in D$ and all $n \times n$ doubly stochastic matrices S. Similarly f is said to be *Schur-concave* if the inequality in (4) is reversed.

Clearly a Schur-convex or Schur-concave function must be symmetric, in other words, $f(P\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in D$, where P is any permutation matrix. From Schur's theorem (Theorem 3.A.4 in [3]), if f is a symmetric and continuously differentiable function of two variables, then f is Schur-convex if and only if

$$(x-y)(f_x(x,y) - f_y(x,y)) \geq 0, \quad \forall\, (x,y) \in D. \tag{5}$$

Similarly, f is Schur-concave if and only the above inequality is reversed.

We recall that a function f defined on a convex set D is *convex* if

$$f(p\mathbf{x} + (1-p)\mathbf{y}) \le pf(\mathbf{x}) + (1-p)f(\mathbf{y}), \ \ \forall\, \mathbf{x}, \mathbf{y} \in D, \ p \in [0,1],$$

and *concave* if the above inequality is reversed. If f is second order continuously differentiable in an n-dimensional open convex domain D, then f is a convex (or concave) function in D if and only if its Hessian matrix is positive (or negative) semi-definite in D (Theorem 3.4.6 in [4]).

A Schur-convex function may not be convex, as the example $f(x,y) = -xy$ shows. Conversely, a convex function may not be Schur-convex. For example, the function $f(x,y) = x^2 + 2y^2$ is convex but not Schur-convex. More properties of Schur-convex functions and their relation to convex functions can be seen from the monograph [3].

Although the area-ratio function r defined by (2) is Schur-concave [5], we show that it is not concave in its domain. A simple computation gives

$$r_{xx}(x,y) = 2(\cos 2x + \cos 2z), \ r_{xy}(x,y) = 2\cos 2z, \ r_{yy}(x,y) = 2(\cos 2y + \cos 2z),$$

and thus

$$\begin{vmatrix} r_{xx}(x,y) & r_{xy}(x,y) \\ r_{xy}(x,y) & r_{yy}(x,y) \end{vmatrix} = 4(\cos 2x \cos 2y + \cos 2y \cos 2z + \cos 2z \cos 2x).$$

If we let $x = y = \pi/2 - \epsilon$ and $z = 2\epsilon$, then

$$\cos 2x \cos 2y + \cos 2y \cos 2z + \cos 2z \cos 2x = \cos^2 2\epsilon - 2\cos 2\epsilon \cos 4\epsilon < 0$$

for $\epsilon > 0$ small enough. This shows that the Hessian matrix of r is not negative semi-definite in I, therefore r is not a concave function.

3 NON-CONVEXITY OF THE DIMENSION FUNCTION

Although an analytic expression of the dimension function $d = d(x, y)$ defined for the family of Sierpiński pedal triangles is not available, the classic implicit function theorem [4] ensures that this function is continuously differentiable in the index domain I. As a matter of fact, the implicit differentiation to the equation (1) gives the first order partial derivatives

$$d_x = \frac{d(\cos^d x \tan x - \cos^d z \tan z)}{A} \tag{6}$$

and

$$d_y = \frac{d(\cos^d y \tan y - \cos^d z \tan z)}{A}, \tag{7}$$

where

$$A = \cos^d x \ln \cos x + \cos^d y \ln \cos y + \cos^d z \ln \cos z.$$

Since the value $d(x, y)$ approaches 2 as the point $(x, y) \in I$ approaches the boundary of the region I, and since

$$\lim_{z \to \frac{\pi}{2}} \cos^d z \tan z = 0$$

and

$$\lim_{z \to \frac{\pi}{2}} \cos^d z \ln \cos z = 0,$$

we can easily see that the function d is continuously differentiable on the closure of I.

Now we calculate the second order partial derivative d_{xx} of the function d with respect to x. From the expression (6),

$$Ad_x = d(\cos^d x \tan x - \cos^d z \tan z).$$

Taking derivative with respect to x gives

$$
\begin{aligned}
& A_x d_x + A d_{xx} \\
= \;& d_x(\cos^d x \tan x - \cos^d z \tan z) + d\left[\cos^d x(d_x \ln\cos x - d\tan x)\tan x\right. \\
& + \; \left.\cos^d x \sec^2 x - \cos^d z(d_x \ln\cos z + d\tan z)\tan z + \cos^d z \sec^2 z\right] \\
= \;& d_x(\cos^d x \tan x - \cos^d z \tan z) + d\left[d_x \cos^d x \tan x \ln\cos x + (1-d)\cos^d x \tan^2 x\right. \\
& + \; \left.\cos^d x - d_x \cos^d z \tan z \ln\cos z + (1-d)\cos^d z \tan^2 z + \cos^d z\right] \\
= \;& d_x\left[\cos^d x \tan x(1 + d\ln\cos x) - \cos^d z \tan z(1 + d\ln\cos z)\right] \\
& + d\left[\cos^d x(1 + (1-d)\tan^2 x) + \cos^d z(1 + (1-d)\tan^2 z)\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
A_x &= d_x(\cos^d x \ln^2\cos x + \cos^d y \ln^2\cos y + \cos^d z \ln^2\cos z) \\
&\quad - \; \cos^d x \tan x(1 + d\ln\cos x) + \cos^d z \tan z(1 + d\ln\cos z),
\end{aligned}
$$

$$
\begin{aligned}
A_x d_x &= d_x^2(\cos^d x \ln^2\cos x + \cos^d y \ln^2\cos y + \cos^d z \ln^2\cos z) \\
&\quad - d_x[\cos^d x \tan x(1 + d\ln\cos x) - \cos^d z \tan z(1 + d\ln\cos z)].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
A d_{xx} &= 2d_x[\cos^d x \tan x(1 + d\ln\cos x) - \cos^d z \tan z(1 + d\ln\cos z)] \\
&\quad + d[\cos^d x(1 + (1-d)\tan^2 x) + \cos^d z(1 + (1-d)\tan^2 z)] \\
&\quad - d_x^2(\cos^d x \ln^2\cos x + \cos^d y \ln^2\cos y + \cos^d z \ln^2\cos z) \\
&= 2d_x[\cos^d x \tan x(1 + d\ln\cos x) - \cos^d z \tan z(1 + d\ln\cos z)] \\
&\quad + d[\cos^{d-2} x(1 - d\sin^2 x) + \cos^{d-2} z(1 - d\sin^2 z)] \\
&\quad - d_x^2(\cos^d x \ln^2\cos x + \cos^d y \ln^2\cos y + \cos^d z \ln^2\cos z) \\
&= \frac{2d(u-v)}{A}(a-b) + d(p+q) - \left[\frac{d(u-v)}{A}\right]^2 B,
\end{aligned}
$$

where

$$u = \cos^d x \tan x, \quad v = \cos^d z \tan z,$$

$$a = u(1 + d \ln \cos x), \quad b = v(1 + d \ln \cos z),$$

$$p = \cos^{d-2} x(1 - d \sin^2 x), \quad q = \cos^{d-2} z(1 - d \sin^2 z),$$

and

$$B = \cos^d x \ln^2 \cos x + \cos^d y \ln^2 \cos y + \cos^d z \ln^2 \cos z.$$

Let $x = z = \pi/4 + \epsilon$ and $y = \pi/2 - 2\epsilon$. Then $a = b$ and $u = v$. Since $d(\pi/4, \pi/2) = 2$, and since $d_x(\pi/4, \pi/2) = 0$ from (6) and $d_y(\pi/4, \pi/2) = 2/\ln 2$ from (7),

$$\begin{aligned} d\left(\frac{\pi}{4} + \epsilon, \frac{\pi}{2} - 2\epsilon\right) &= d\left(\frac{\pi}{4}, \frac{\pi}{2}\right) + d_x\left(\frac{\pi}{4}, \frac{\pi}{2}\right)\epsilon - 2d_y\left(\frac{\pi}{4}, \frac{\pi}{2}\right)\epsilon + O(\epsilon^2) \\ &= 2 - \frac{4}{\ln 2}\epsilon + O(\epsilon^2) \end{aligned}$$

for small $\epsilon > 0$. Now, since

$$\begin{aligned} & 1 - d\sin^2\left(\frac{\pi}{4} + \epsilon\right) = 1 - \frac{d}{2}(1 + \sin 2\epsilon) \\ = \;& 1 - \frac{1}{2}\left[2 - \frac{4}{\ln 2}\epsilon + O(\epsilon^2)\right]\left[1 + 2\epsilon + O(\epsilon^2)\right] \\ = \;& 1 - \left[1 + 2\epsilon - \frac{2}{\ln 2}\epsilon + O(\epsilon^2)\right] = \left(\frac{2}{\ln 2} - 2\right)\epsilon + O(\epsilon^2) > 0 \end{aligned}$$

for $\epsilon > 0$ small enough,

$$p = q = \cos^{d-2}\left(\frac{\pi}{4} + \epsilon\right)\left[1 - d\sin^2\left(\frac{\pi}{4} + \epsilon\right)\right] > 0,$$

which ensures that, since $A < 0$ everywhere,

$$d_{xx}\left(\frac{\pi}{4} + \epsilon, \frac{\pi}{2} - 2\epsilon\right) = \frac{2dp}{A} < 0.$$

This shows that the Hessian matrix of d is not positive semi-definite in I, so the dimension function d is not convex on its domain.

In the following we further show that the dimension function d is also not Schur-convex in I, thus disproving the Schur-convexity conjecture of [5]. Since d is symmetric and continuously differentiable in the open set I, by (5), d is a Schur-convex if and only if

$$(x-y)[d_x(x,y)-d_y(x,y)] \geq 0, \ \ \forall\ (x,y) \in I.$$

From (6) and (7),

$$A(x-y)[d_x(x,y)-d_y(x,y)] = (x-y)(\cos^d x \tan x - \cos^d y \tan y).$$

Since $A<0$, in order to disprove the conjecture, it is equivalent to show that there is a point $(x_0,y_0)\in I$ with $x_0>y_0$ such that

$$\cos^d x_0 \tan x_0 > \cos^d y_0 \tan y_0. \tag{8}$$

Let $\epsilon>0$ be small and $x=\pi/4+\epsilon, y=\pi/4$. Then $d(\pi/4,\pi/4)=2$ and $d_x(\pi/4,\pi/4)=-2/\ln 2$ by (6), so

$$d\left(\frac{\pi}{4}+\epsilon,\frac{\pi}{4}\right) = d\left(\frac{\pi}{4},\frac{\pi}{4}\right) + d_x\left(\frac{\pi}{4},\frac{\pi}{4}\right)\epsilon + O(\epsilon^2) = 2-\frac{2}{\ln 2}\epsilon + O(\epsilon^2).$$

It follows that

$$\begin{aligned} \cos^d x \tan x &= \cos^{d-1} x \sin x = \left[\frac{\sqrt{2}}{2}(\cos\epsilon - \sin\epsilon)\right]^{d-1} \frac{\sqrt{2}}{2}(\cos\epsilon+\sin\epsilon) \\ &= \left(\frac{\sqrt{2}}{2}\right)^d (\cos\epsilon-\sin\epsilon)^{d-1}(\cos\epsilon+\sin\epsilon) \end{aligned}$$

and

$$\cos^d y \tan y = \left(\frac{\sqrt{2}}{2}\right)^d.$$

Since

$$
\begin{aligned}
& (\cos\epsilon - \sin\epsilon)^{d-1}(\cos\epsilon + \sin\epsilon) \\
= \; & \left[1 - \epsilon - \frac{\epsilon^2}{2} + O(\epsilon^3)\right]^{1-\frac{2}{\ln 2}\epsilon + O(\epsilon^2)} \left[1 + \epsilon - \frac{\epsilon^2}{2} + O(\epsilon^3)\right] \\
= \; & 1 + \left(\frac{2}{\ln 2} - 2\right)\epsilon^2 + O(\epsilon^3) > 1
\end{aligned}
$$

for ϵ small enough, (8) is satisfied by a point $(x_0, y_0) \in I$ with $x_0 = \pi/4 + \epsilon_0$ and $y_0 = \pi/4$ for some small positive number ϵ_0.

In summary, we have proved the following assertion.

Theorem 3.1 *The dimension function d for the Sierpiński pedal triangles is neither Schur-convex nor convex in its whole domain I.*

4 CONCLUSIONS

In this paper we disproved the last conjecture of the paper [5] that the dimension function d for the Sierpiński pedal triangles be Schur-convex in its domain. However, since $d_{xx}(\pi/3, \pi/3) = 2(\ln 3/\ln 2 - 4/3) > 0$ and

$$
\begin{vmatrix} d_{xx}(x,y) & d_{xy}(x,y) \\ d_{xy}(x,y) & d_{yy}(x,y) \end{vmatrix} \left(\frac{\pi}{3}, \frac{\pi}{3}\right) = 3\left(\frac{\ln 3}{\ln 2} - \frac{4}{3}\right)^2 \left(\frac{\ln 3}{\ln 4}\right)^2 > 0,
$$

d is convex in a neighborhood of its global minima $(\pi/3, \pi/3)$. It would be an interesting problem to find the maximal subregion Ω of the index domain I on which the dimension function d is convex and characterize the boundary of Ω. Based on our above analysis and solution to the conjecture of [5], we propose the following new conjecture: The dimension function d is both Schur-convex and convex everywhere in I except for the points near the boundary of I.

5 ACKNOWLEDGEMENTS

This research is supported by Morningside Center of Mathematics, Chinese Academy of Sciences, and by National Natural Science Foundation of China (Grant Nos. 10872037 and 60931002).

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